Math 115A
Homework 1 Solutions

April 27, 2006

1. (a) The set
$$S = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{Z} \right\}$$
is closed under addition because the integers are closed under addition.
S is not closed under multiplication, since $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S$, but
$$\frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \notin S$$

(b) The set
$$S = \text{span}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \cup \text{span}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} \right)$$
is closed under scalar multiplication since for any $v \in S$,
$$v = c \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ or } v = c \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
so
$$rv = rc \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in S \text{ or } rv = rc \begin{pmatrix} -1 \\ 1 \end{pmatrix} \in S$$
S is not closed under addition since
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \notin S$$

2. (i) For any $x, y \in V$,
$$x \oplus y = xy = yx = y \oplus x$$

(ii) For any $x, y, z \in V$,

$$x \oplus (y \oplus z) = x \oplus (yz) = xyz = (xy) \oplus z = (x \oplus y) \oplus z$$

(iii) For any $x \in V$,

$$1 \oplus x = 1x = x$$

Thus 1 is the zero vector.

(iv) Remember from (iii), 1 is the zero vector. Then for any x, we have

$$x \oplus \frac{1}{x} = x\frac{1}{x} = 1$$

Since x is a positive-real number, $\frac{1}{x}$ is also a positive real number.

(v) For any $x \in V$,

$$1 \otimes x = x^1 = x$$

(vi) For any $a, b \in \mathbb{R}$, and $x \in V$,

$$a \otimes (b \otimes x) = a \otimes x^b = \left(x^b\right)^a = x^{ba} = x^{ab} = (ab) \otimes x$$

(vii) For any $a \in \mathbb{R}$, $x, y \in V$,

$$a \otimes (x \oplus y) = a \otimes (xy) = (xy)^a = x^a y^a = (a \otimes x) \oplus (a \otimes y)$$

(viii) For any $a, b \in \mathbb{R}$, $x \in V$,

$$(a + b) \otimes x = x^{a+b} = x^a x^b = x^a \oplus x^b = (a \otimes x) \oplus (b \otimes x)$$

3. This is not a vector space since axiom (i) is not satisfied

$$(x_1, x_2)+(y_1, y_2) = (x_1+y_2, x_2+y_1) \neq (y_1+x_2, y_2+x_1) = (y_1, y_2)+(x_1, x_2)$$

Many of the other axioms are not satisfied, but we can stop after any one of them is not satisfied.

4. (a) Let $W = \{(b_1, b_2, b_3) : b_1 = 0\}$. Then W is a subspace

(i)

$$(0, 0, 0) \in W$$

(ii)

$$(0, b_2, b_3) + (0, b_2', b_3') = (0, b_2 + b_2', b_3 + b_3') \in W$$

So W is closed under vector addition.

(iii)

$$c(0, b_2, b_3) = (0, cb_2, cb_3) \in W$$

So W is closed under scalar multiplication.

(b) Let $W = \{(b_1, b_2, b_3) : b_1 = 1\}$. Then W is not a subspace

(i)

$$(0, 0, 0) \notin W$$

In fact, W is not closed under either vector addition, or scalar multiplication, but we can stop after one axiom is not satisfied.

(c) Let $W = \{(b_1, b_2, b_3) : b_2 b_3 = 0\}$. Then W is not a subspace

(i)

$$(0, 0, 0) \in W$$

Since $0 \cdot 0 = 0$.

(ii)

$$(0, 1, 0) + (0, 0, 1) = (0, 1, 1) \notin W$$

So W is not closed under vector addition.

(d) Let $W = \{(b_1, b_2, b_3) : b_3 - b_2 + 3b_1 = 0\}$. Then W is a subspace. Ideally you remember from Math 33A that this is the equation for the plane perpendicular to the vector $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$.

(i)

$$(0, 0, 0) \in W$$

Since $0 - 0 + 3 \cdot 0 = 0$.

(ii) If $(x_1, y_1, z_1), (x_2, y_2, z_2) \in W$, then

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

and

$$(x_1+x_2)-(y_1+y_2)+3(z_1+z_2) = (x_1-y_1+3z_1)+(x_2-y_2+3z_2) = 0+0$$

So W is closed under vector addition.

(iii) If $(x_1, y_1, z_1) \in W$, then

$$c(x_1, y_1, z_1) = (cx_1, cy_1, cz_1) \in W$$

and

$$cx_1 - cy_1 + 3cz_1 = c(x_1 - y_1 + 3z_1) = c \cdot 0 = 0$$

So W is closed under scalar multiplication.

5. A subspace is closed under vector addition and scalar multiplication, so if W is a subspace containing A and B, then W contains $A - B = I$.

6. (i) We know $\{0\}$ is a subspace of every vector space, so $\{0\}$ is a subspace of $\mathbb{R}$.
   Let W be a nonzero subspace of $\mathbb{R}$. Then there is some $0 \neq a \in W$. For any $r \in \mathbb{R}$, we have that $r = \frac{r}{a}a$. Thus r is a scalar multiple of a. Since W is closed under scalar multiplication, $r \in W$. Since r was aribtrary, we conclude that $W = \mathbb{R}$. Thus there are exactly 2 subspaces of $\mathbb{R}$,

$$\{0\}, \mathbb{R}$$

   (ii) We know $\{0\}$ is a subspace of $\mathbb{R}^2$.
   For any nonzero vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$, we know that $\text{span}(\{v\}) \subsetneq \mathbb{R}^2$, and $\text{span}(\{v\})$ is a subspace. In fact, $\text{span}(\{v\})$ is the line through the origin and the point $\begin{pmatrix} a \\ b \end{pmatrix}$. There are an infinite number of lines through the origin, so there are an infinite number of subspaces of $\mathbb{R}^2$.
   For the sake of completeness, we mention that the only subspaces of $\mathbb{R}^2$, are $\{0\}$, lines through the origin, and $\mathbb{R}^2$.