

## Homework 8

June 13, 2006

1.

$$\begin{aligned}\left\langle \begin{pmatrix} 2 \\ 1+i \\ i \end{pmatrix}, \begin{pmatrix} 2-i \\ 2 \\ 1+2i \end{pmatrix} \right\rangle &= 2\overline{(2-i)} + (1+i)\overline{2} + i\overline{(1+2i)} \\ &= 2(2+i) + (1+i)(2) + i(1-2i) \\ &= 4 + 2i + 2 + 2i + i + 2 \\ &= 8 + 5i\end{aligned}$$

And

$$\begin{aligned}\|u\| &= \sqrt{\left\langle \begin{pmatrix} 2 \\ 1+i \\ i \end{pmatrix}, \begin{pmatrix} 2 \\ 1+i \\ i \end{pmatrix} \right\rangle} \\ &= \sqrt{2\overline{2} + (1+i)\overline{(1+i)} + i\overline{i}} \\ &= \sqrt{4 + 2 + 1} \\ &= \sqrt{7}\end{aligned}$$

And

$$\begin{aligned}\|v\| &= \sqrt{\left\langle \begin{pmatrix} 2-i \\ 2 \\ 1+2i \end{pmatrix}, \begin{pmatrix} 2-i \\ 2 \\ 1+2i \end{pmatrix} \right\rangle} \\ &= \sqrt{(2-i)\overline{(2-i)} + (2)\overline{2} + (1+2i)\overline{(1+2i)}} \\ &= \sqrt{5 + 4 + 5} \\ &= \sqrt{14}\end{aligned}$$

And

$$\begin{aligned}\|u + v\| &= \sqrt{\left\langle \begin{pmatrix} 4 - i \\ 3 + i \\ 1 + 3i \end{pmatrix}, \begin{pmatrix} 4 - i \\ 3 + i \\ 1 + 3i \end{pmatrix} \right\rangle} \\ &= \sqrt{(4 - i)(4 - i) + (3 + i)(3 + i) + (1 + 3i)(1 + 3i)} \\ &= \sqrt{17 + 10 + 10} \\ &= \sqrt{37}\end{aligned}$$

The Cauchy-Schwartz inequality says

$$\sqrt{89} = |\langle u, v \rangle| \leq \|u\| \|v\| = \sqrt{7} \cdot \sqrt{14} = \sqrt{98}$$

The triangle inequality says

$$6.08 \approx \sqrt{37} = \|u + v\| \leq \|u\| + \|v\| = \sqrt{7} + \sqrt{14} \approx 6.39$$

2.

$$\langle f, g \rangle = \int_0^1 x e^x dx = (x - 1)e^x \Big|_0^1 = 1$$

And

$$\|f\|^2 = \int_0^1 x^2 dx = \frac{1}{3}$$

$$\text{so } \|f\| = \sqrt{\frac{1}{3}}.$$

$$\|g\|^2 = \int_0^1 e^{2x} dx = \frac{1}{2} e^{2x} \Big|_0^1 = \frac{e^2 - 1}{2}$$

$$\text{so } \|g\| = \sqrt{\frac{e^2 - 1}{2}}.$$

$$\|f + g\|^2 = \int_0^1 (x + e^x)(x + e^x) dx = \frac{1}{3} + \frac{e^2 - 1}{2} + 2 = \frac{3e^2 + 11}{6}$$

$$\text{so } \|f + g\| = \sqrt{\frac{3e^2 + 11}{6}}.$$

The Cauchy-Schwarz inequality says

$$1 = |\langle f, g \rangle| \leq \|f\| \|g\| = \sqrt{\frac{1}{3}} \sqrt{\frac{e^2 - 1}{2}} \approx 1.032$$

The Triangle Inequality says

$$2.3511 \approx \sqrt{\frac{3e^2 + 11}{6}} = \|f + g\| \leq \|f\| + \|g\| = \sqrt{\frac{1}{3}} + \sqrt{\frac{e^2 - 1}{2}} \approx 2.365$$

3. These follow directly from the triangle inequality:  $\|u + v\| \leq \|u\| + \|v\|$

(a)

$$\|u - v\| = \|u + (-v)\| \leq \|u\| + \|-v\| = \|u\| + \|v\|$$

(b)

$$\|u\| = \|(u + v) - v\| \leq \|u + v\| + \|v\|$$

Then subtracting  $\|v\|$  from both sides gives

$$\|u\| - \|v\| \leq \|u + v\|$$

(c) Notice the absolute value on the right hand side of the problem statement.

$$\|u\| = \|(u - v) + v\| \leq \|u - v\| + \|v\|$$

Then subtracting  $\|v\|$  from both sides gives

$$\|u\| - \|v\| \leq \|u - v\|$$

Now performing the exact same trick with the roles of  $u, v$  reversed, we get

$$\|v\| - \|u\| \leq \|v - u\| = \|u - v\|$$

Combining these two inequalities gives

$$\| \|u\| - \|v\| \| \leq \|u - v\|$$

4. If  $u, v$  are orthogonal this means  $\langle u, v \rangle = \langle v, u \rangle = 0$ . Thus

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \langle u, u \rangle + \langle v, v \rangle \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

The picture in  $\mathbb{R}^2$  should just be a right triangle. That is to help you recognize that in  $\mathbb{R}^2$  this is just the Pythagorean Theorem.

5. We have

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$$

So

$$u_1 = \frac{1}{\|v_1\|} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$u'_2 = v_2 - \langle v_2, u_1 \rangle u_1 = v_2 - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

$$u_2 = \frac{u'_2}{\|u'_2\|} = \sqrt{\frac{2}{3}} \begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

$$u'_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2 = v_3 - 2\sqrt{2}u_1 - \frac{2\sqrt{6}}{3}u_2 = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$u_3 = \frac{u'_3}{\|u'_3\|} = \frac{u'_3}{\sqrt{3}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

To calculate  $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$  in this basis we could use the methods we've always used, but since this basis is orthonormal, there is a shortcut.

If we call  $v = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ , then in this basis it is

$$\begin{pmatrix} \langle v, u_1 \rangle \\ \langle v, u_2 \rangle \\ \langle v, u_3 \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ \sqrt{3} \\ 0 \end{pmatrix}$$